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On Numerical Range of Multiplication Operator

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Abstract

Let H be an infinite dimensional complex Hilbert space and $A, B \in B(H)$ where $B(H)$ is the C^* -algebra of all bounded linear operators on H . Let $M_{AB} : B(H) \rightarrow B(H)$ be a multiplication operator induced by A and B defined by

$$M_{AB}(X) = AXB \tag{1}$$

In this paper we show that the numerical range of multiplication operator is given by

$$V(M_{AB/B(B(H))}) = [\bigcup_{U \in U(H)} W(U^* A U B)^-]^- \text{ for all } A, B \in B(H)$$

and U a unitary operator on the algebra $B(H)$ where V is the algebraic numerical range and W is the classical numerical range. The results obtained are an extension of the the work done by Barraa [4].

Mathematics Subject Classification: Primary 47A12, Secondary 47A30, 47B47

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1. Introduction

Multiplication operator

There are various settings for the definition of multiplication operator.

Let \mathcal{A} be a unital Banach algebra. The multiplication operator $M_{a,b} : \mathcal{A} \rightarrow \mathcal{A}$ is defined by

$$M_{ab}(x) = axb \tag{2}$$

where $x \in \mathcal{A}$ and $a, b \in \mathcal{A}$ are fixed.

Numerical range

For operators in a Hilbert space H , the notion of numerical range (or field of values) is important in various applications in the study of operators. The numerical range of an operator $T \in B(H)$ is a subset of a complex plane \mathbb{C} defined by

$W(T) = \{\langle Tx, x \rangle : \|x\| = 1\}$. This set is convex but not closed in general. For the multiplication operator we have

$$W(M_{A,B}) = \{\langle AXBx, x \rangle : x \in H, \|x\| \leq 1\} \quad (3)$$

The algebraic numerical range of $a \in \mathcal{A}$ for a unital C^* -algebra \mathcal{A} is defined by;

$$V(a/\mathcal{A}) = \{f(a) : f \in \mathcal{A}^*, \|f\| = 1 = f(e)\} \quad (4)$$

which is a closed convex set. Similarly,

$$V(M_{AB/B(H)}) = \{f(M_{AB}) : f \in B(B(H))^*, \|f\| = 1 = f(AB)\} \quad (5).$$

If $\mathcal{A} = B(X)$ where $B(X)$ is the algebra of bounded linear operators on a normed space X and $T \in B(X)$ then we have the spatial numerical range of T defined by;

$$V_o(T) = \{f(Tx) : x \in X, f \in X^* \text{ with } \|f\| = \|x\| = 1 = f(x)\}. \quad (6).$$

Theorem 1

Let H be a Hilbert space and $B(H)$ the algebra of all bounded linear operators on H . Then the numerical range $V(M_{AB})$ is a convex set.

Proof

We need to show that if $\alpha_1, \alpha_2 \in V(M_{AB})$ and $t \in (0, 1)$ then

$$\alpha = t\alpha_1 + (1 - t)\alpha_2 \in V(M_{AB}).$$

Let $\alpha_1, \alpha_2 \in V(M_{AB})$ then there exists states f_1 and f_2 such that

$$\alpha_1 = f_1(M_{AB}) \text{ and } \alpha_2 = f_2(M_{AB}) \text{ where } M_{AB} \in B(B(H)),$$

$$f_1(AB) = 1 = \|f_1\| \text{ and } f_2(AB) = 1 = \|f_2\|. \text{ We define } f \text{ on } B(B(H)) \text{ by}$$

$f(M_{AB}) = tf_1(M_{AB}) + (1 - t)f_2(M_{AB})$. It suffices to show that f is a state by showing that it is linear, positive and its norm is one.

f is linear

Let $\mu_1, \mu_2 \in \mathbb{C}$ and $(M_{AB}) \in B(B(H))$ then

$$\begin{aligned} f(\mu_1(M_{AB}) + \mu_2(M_{AB})) &= tf_1(\mu_1(M_{AB}) + \mu_2(M_{AB})) + (1 - t)f_2(\mu_1(M_{A,B}) + \mu_2(M_{AB})) \\ &= \{tf_1(\mu_1(M_{AB})) + (1-t)f_2(\mu_1(M_{AB}))\} + \{tf_1(\mu_2(M_{AB})) + (1-t)f_2(\mu_2(M_{A,B}))\} \\ &= \{\mu_1(tf_1(M_{AB})) + \mu_1((1-t)f_2(M_{AB}))\} + \{\mu_2(tf_1(M_{AB}) + \mu_2((1-t)f_2(M_{AB}))\} \\ &= \mu_1\{tf_1(M_{AB}) + (1 - t)f_2(M_{AB})\} + \mu_2\{tf_1(M_{AB}) + (1 - t)f_2(M_{AB})\} \\ &= \mu_1(f(M_{AB})) + \mu_2(f(M_{AB})) \end{aligned}$$

f is positive

$$\begin{aligned}
f((AXB)^*AXB) &= tf_1((AXB)^*AXB) + (1-t)f_2((AXB)^*AXB) \geq 0 \text{ Since} \\
f_1((AXB)^*AXB) &= ((AXB)^*AXBx, x) \\
&= (AXBx, AXBx) = \|AXB\|^2 = \|A\|^2\|B\|^2 \geq 0.
\end{aligned}$$

Now,

$$f(AB) = tf_1(AB) + (1-t)f_2(AB) = t + (1-t) = 1.$$

$1 = |f(AB)| \leq \|f\|\|1\| = \|f\|$ implying that $\|f\| \geq 1$. Also,

$$\begin{aligned}
|f(AB)| &= |tf_1(AB) + (1-t)f_2(AB)| \\
&\leq |tf_1(AB)| + |(1-t)f_2(AB)| \\
&\leq |t|\|f_1\| + |1-t|\|f_2\| = 1. \text{ Implying that } \|f\| \leq 1. \text{ Thus } f \text{ is a state in} \\
&B(B(H))^*. \text{ Therefore, } f(M_{AB}) \in V(M_{AB}) \text{ so } V(M_{AB}) \text{ is convex.}
\end{aligned}$$

Theorem 2

$$V(M_{AB/B(B(H))}) = \overline{W(AXB)}$$

Proof

Here, it will be shown that $V(M_{AB}) \subseteq \overline{W(AXB)}$ and $\overline{W(AXB)} \subseteq V(M_{AB})$.

Let $\alpha \in \overline{W(AXB)}$ then there exists a sequence $\{x_n\}_{n \geq 1}$ of unit vectors in H such that $\lim_{n \rightarrow \infty} \langle AXBx_n, x_n \rangle = \alpha$ and $\lim_{n \rightarrow \infty} \|AXBx_n\| = \|AXB\|$ for all $M_{AB} \in B(B(H))$.

We define a functional f on $B(B(H))$ by $f(AXB) = \lim_{n \rightarrow \infty} \langle AXBx_n, x_n \rangle$ so that $f(AXB) = \alpha$

We will show that f is a state.

First, f is linear since if $(AXB) \in B(B(H))$ and $\lambda, \mu \in \mathbb{C}$ then,

$$\begin{aligned}
f(\lambda(AXB) + \mu(AXB)) &= \lim_{n \rightarrow \infty} \langle (\lambda(AXB) + \mu(AXB))x_n, x_n \rangle \\
&= \lim_{n \rightarrow \infty} \langle \lambda(AXB)x_n, x_n \rangle + \lim_{n \rightarrow \infty} \langle \mu(AXB)x_n, x_n \rangle \\
&= \lambda \lim_{n \rightarrow \infty} \langle (AXB)x_n, x_n \rangle + \mu \lim_{n \rightarrow \infty} \langle (AXB)x_n, x_n \rangle \\
&= \lambda f(AXB) + \mu f(AXB).
\end{aligned}$$

Also, f is positive since

$$\begin{aligned}
f(AXB)^*(AXB) &= \lim_{n \rightarrow \infty} \langle ((AXB)^*AXB)x_n, x_n \rangle \\
&= \lim_{n \rightarrow \infty} \langle AXBx_n, AXBx_n \rangle \\
&= \{\lim_{n \rightarrow \infty} \|AXBx_n\|\}^2 = \|AXB\|^2 \geq 0.
\end{aligned}$$

Finally, $|f(AXB)| = |\lim_{n \rightarrow \infty} \langle AXBx_n, x_n \rangle|$

$$\leq \lim_{n \rightarrow \infty} \|AXBx_n\| \lim_{n \rightarrow \infty} \|x_n\| = \|AXB\|.$$

Thus $\|f\| \leq 1$.

Now, $1 = \|f(AB)\| \leq \|f\| \|AB\| = \|f\|$ so, that $\|f\| \geq 1$. Therefore $\|f\| = 1$ and so $\alpha \in V(M_{AB})$. Hence $\overline{W(AXB)} \subseteq V(M_{AB})$

Next we show that $V(AXB) \subseteq \overline{W(AXB)}$ See[1].

Let $\lambda \in V(AXB)$ and λ not in $W(AXB)$ and deduce a contradiction. Therefore, there exists a state $f \in B(B(H))^*$ such that $f(AXB) = \lambda$ and $f((AXB)^*AXB) \geq 0$. Since $W(AXB)$ is convex, then by rotating M_{AB} , we may assume that $Re W(AXB) \leq Re \lambda - \alpha, \alpha > 0$.

Let $G = \{x \in H : \|x\| = 1 \text{ and } Re \langle AXBx, x \rangle \geq Re \lambda - \frac{\alpha}{2}, \alpha > 0\}$ and $\vartheta = \sup\{\|AXBx\| : x \in H\}$. Then $\vartheta < 0$.

The set G is nonempty because if it is not, then for all $x \in H, \|x\| = 1$ we shall have

$$Re \langle AXBx, x \rangle < Re \lambda - \frac{\alpha}{2}, \alpha > 0.$$

But since f is a weak*-limit of convex combinations of vector states for all $\varepsilon > 0$ there exists $N = N(\varepsilon)$ such that for all $n > N, |f_n(AXB) - f(AXB)| < \varepsilon$.

Also we can find $M = M(\varepsilon)$ such that for all $n > M,$

$$|f_n((AXB)^*AXB) - f((AXB)^*AXB)| < \varepsilon.$$

Taking $\varepsilon < \frac{\alpha}{2}$ and $n > \max(N, M)$ and since

$$f_n(AXB) = \sum_{i=1}^n \alpha_i \omega_{x_i}(AXB) = \sum_{i=1}^n \alpha_i \langle AXBx_i, x_i \rangle \text{ for } 0 \leq \alpha_i \leq 1 \text{ and } \sum_{i=1}^n \alpha_i = 1 \text{ we have}$$

$$\begin{aligned} Re f_n(x) &= Re \sum_{i=1}^n \alpha_i \omega_{x_i}(AXB) = Re \sum_{i=1}^n \alpha_i \langle AXBx_i, x_i \rangle \\ &= \sum_{i=1}^n \alpha_i Re \langle AXBx_i, x_i \rangle \leq Re \lambda - \frac{\alpha}{2}. \end{aligned}$$

But $f_n(x) > f(AXB) - \varepsilon$ and therefore $Re f_n(x) > Re \lambda - \varepsilon$ which implies that $\varepsilon > \frac{\alpha}{2}$. This is a contradiction.

Now, for all $x_i \in G$, we have that $\|AXBx_i\| \leq \vartheta$.

Since $f((AXB)^*AXB) < f_n((AXB)^*AXB) + \varepsilon$ and $0 \leq f((AXB)^*AXB)$ we obtain

$$0 \leq f((AXB)^*AXB) < f_n((AXB)^*AXB) + \varepsilon = \sum_{i=1}^n \|AXBx_i\|^2 + \varepsilon < \vartheta^2 < 0$$

which is also a contradiction. Thus λ not in $W(AXB)$ implies that λ is not in $V(AXB)$. Hence $\lambda \in V(AXB)$ implies that $\lambda \in W(AXB)$ and so $V(AXB) \subseteq W(AXB)$ and since $W(AXB)$ is convex, then $V(AXB) \subseteq \overline{W(AXB)}$.

Main result

Theorem 3

Let H be a complex Hilbert space and $B(H)$ a C^* -algebra of all bounded linear operators on H . Then,

$$V(M_{AB/B(B(H))}) = [\bigcup_{U \in U(H)} W(U^* A U B)^-]^-.$$

For all $A, B \in B(H)$ and U a unitary operator.

To prove this theorem we use the following Lemma 4

Lemma 4

Let A and B be elements in $B(H)$. Then, $W(AB) \subset V(M_{AB/B(B(H))})$ where $W(AB) = \{\langle ABx, x \rangle\}$.

Proof

Let $\alpha \in W(AB)$ then by definition of the classical numerical range, there exist $x \in H$ with $\|x\| = 1$ such that;

$$\alpha = \langle ABx, x \rangle = \text{tr}(AB(x \otimes x)) \text{ where } \text{tr}(\cdot) \text{ is a linear form trace.}$$

We denote this linear form by $\Psi_{x \otimes x}$ and define it as

$\Psi_{x \otimes x}(X) = \text{tr}(X(x \otimes x)) = \langle Xx, x \rangle$ on $B(B(H))$. The linear form is bounded and its norm is equal to one that is;

$$\|\Psi_{x \otimes x}\| = \|x \otimes x\| = 1.$$

The form $\Psi_{x \otimes x}$ is also a state since

$$\Psi_{x \otimes x}(I) = \text{tr}(x \otimes x) = \langle x, x \rangle = \|x\|^2 = 1 \text{ and}$$

$$\Psi_{x \otimes x} X^* X = \text{tr}(X^* X(x \otimes x)) = \langle X^* Xx, x \rangle = \langle Xx, Xx \rangle = \|Xx\|^2 \geq 0.$$

So $\Psi_{x \otimes x}(M_{AB}(I_H)) \subset V(M_{AB/B(B(H))})$ and we have that

$$\Psi_{x \otimes x}(M_{AB}(I_H)) = \Psi_{x \otimes x}(AB) = \text{tr}(AB(x \otimes x)) = \langle ABx, x \rangle = \alpha.$$

Thus $W(AB) \in W(M_{AB}) \subset V(M_{AB/B(B(H))})$.

Let E be a Banach space. Then $T \in B(E)$ is said to be an isometry if $\|Tx\| = \|x\|$ for all $x \in E$. If T is an invertible isometry, then its inverse T^{-1} is also an isometry therefore,

$$V(TST^{-1} /_{B(E)}) = V(T^{-1}ST /_{B(E)}) = V(S /_{B(E)}) \quad (7)$$

for all $S \in B(E)$.

If $E = H$ then $T = U$ and $T^{-1} = U^*$. Thus from equation (7) we have that

$$V(UAU^* /_{B(H)}) = V(U^*AU /_{B(H)}) = V(A /_{B(H)}) \tag{8}$$

for all $A \in B(H)$.

Given two isometries $U, V \in H$, then

$$V(M_{U^*AU V^*BV /_{B(B(H))}}) = V(M_{AB /_{B(B(H))}}) \tag{9}$$

for all $A, B \in B(H)$.

Now, taking an invertible isometry R_{UV^*} with R_{U^*V} as its inverse, then

$$V(M_{U^*AU V^*BV /_{B(B(H))}}) = V(R_{UV^*} M_{AB} R_{U^*V} /_{B(B(H))}),$$

and by lemma 4 $W(U^*AU V^*BV) \subset V(R_{UV^*} M_{AB} R_{U^*V} /_{B(B(H))})$ and

$$\bigcup_{U, V \in U(H)} W(U^*AU V^*BV) \subset V(M_{AB /_{B(B(H))}}).$$

Since the numerical range is closed and the product of two unitaries is also a unitary, then

$$[\bigcup_{U \in U(H)} W(U^*AUB)^-]^- \subset V(M_{AB /_{B(B(H))}}) \text{ or}$$

$$[\bigcup_{V \in U(H)} W(V^*AVB)^-]^- \subset V(M_{AB /_{B(B(H))}}).$$

Next we proceed to show the inclusion

$$V(M_{AB /_{B(B(H))}}) \subset [\bigcup_{U \in U(H)} W(U^*AUB)^-]^-.$$

Let \mathcal{A} be a Banach algebra, then for any $a \in \mathcal{A}$;

$$V(a /_{\mathcal{A}}) = \bigcap_{z \in \mathbb{C}} \{\lambda : |\lambda - z| \leq \|a - z\|\}. \text{(See [15]).}$$

The norm of multiplication operator is defined by;

$$\begin{aligned} \|M_{AB}\| &= \text{Sup}\{\|M_{AB}(X)\| : \|X\| = 1\} \\ &= \text{Sup}\{\|AXB\| : X \in B(H), \|X\| \leq 1\}. \end{aligned}$$

Theorem 5

Let \mathcal{A} be C^* -algebra, then

$$\begin{aligned} \|M_{AB}\| &= \text{Sup}\{\|M_{AB}(U)\| : U \in U(\mathcal{A})\} \\ &= \text{Sup}\{\|AUB\| : U \in U(\mathcal{A})\} \end{aligned}$$

where $U(\mathcal{A})$ denotes the set of unitaries in \mathcal{A} . (see [11]).

Now, if $\mathcal{A} = B(H)$ then $M_{AB}(U) = AUB$ for all $U \in U(H)$. Therefore,

$$V(M_{AB /_{B(B(H))}}) = \bigcap_{z \in \mathbb{C}} \{\lambda : |\lambda - z| \leq \|M_{A,B} - z\|\}. \text{ But}$$

$$\begin{aligned} \|M_{AB} - z\| &= \text{Sup}\{\|(M_{AB} - z)(U)\| : U \in U(H)\} \\ &= \text{Sup}\{\|(AUB - z)U\| : U \in U(H)\}. \end{aligned}$$

Since the unitary $U \in U(H)$ is an isometry, then

$$\|M_{AB} - z\| = \text{Sup}\{\|U^*AUB - zI_H\| : U \in U(H)\}.$$

So if $\mu \in V(M_{AB/B(B(H))})$ then for all $z \in \mathbb{C}$,

$$\mu \in \{|\lambda - z| \leq \|M_{AB} - zI_{H/B(H)}\|\}.$$

Taking a fixed $\varepsilon > 0$, there exists U_ε such that

$$\|M_{AB} - zI_{H/B(H)}\| < \|U_\varepsilon^*AU_\varepsilon B - zI_H\| + \varepsilon \text{ and by theorem 1.4 we have that,}$$

$$W(U_\varepsilon^*AU_\varepsilon)^- = V(U_\varepsilon^*AU_\varepsilon B)$$

$$= \bigcap_{z \in \mathbb{C}} \{\lambda : |\lambda - z| \leq \|U_\varepsilon^*AU_\varepsilon B - zI_H\|\}$$

and so there exists $\lambda \in W(U_\varepsilon^*AUB)$ such that $|\mu - \lambda| \leq \varepsilon$.

Since ε is arbitrary, $\mu \in [\bigcup_{U \in U(H)} W(U^*AUB)^-]^-$. □

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